A SUPERCRITICAL ELLIPTIC PROBLEM IN A CYLINDRICAL SHELL

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Abstract. We consider the problem

$$-\Delta u = |u|^{p-2}\, u \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega,$$

where $\Omega := \{(y,z) \in \mathbb{R}^{m+1} \times \mathbb{R}^{N-m-1} : 0 < a < |y| < b < \infty\}$, $0 \leq m \leq N-1$ and $N \geq 2$. Let $2^*_{N,m} := 2(N-m)/(N-m-2)$ if $m < N-2$ and $2^*_{N,m} := \infty$ if $m = N-2$ or $N-1$. We show that $2^*_{N,m}$ is the true critical exponent for this problem, and that there exist nontrivial solutions if $2 < p < 2^*_{N,m}$ but there are no such solutions if $p \geq 2^*_{N,m}$.

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To Bernhard Ruf on his birthday, with our friendship and great esteem.

1. Introduction

Consider the Lane-Emden-Fowler problem

$$-\Delta u = |u|^{p-2}\, u \ \text{ in } \mathcal{D}, \qquad u = 0 \ \text{ on } \partial\mathcal{D}, \tag{1.1}$$

where $\mathcal{D}$ is a smooth domain in $\mathbb{R}^N$ and $p > 2$.

If $\mathcal{D}$ is bounded it is well-known that this problem has at least one positive solution and infinitely many sign changing solutions when p is smaller than the critical Sobolev exponent 2^*, defined as $2^* := \frac{2N}{N-2}$ if $N \geq 3$ and as $2^* := \infty$ if $N = 1$ or 2. In contrast, the existence of solutions for $p \geq 2^*$ is a delicate issue. Pohozhaev's identity [12] implies that problem (1.1) has no nontrivial solution if the domain $\mathcal{D}$ is strictly starshaped. On the other hand, Bahri and Coron [2] proved that a positive solution to (1.1) exists if $p = 2^*$ and $\mathcal{D}$ is bounded and has nontrivial reduced homology with $\mathbb{Z}/2$ coefficients.

One may ask whether this last statement is also true for $p > 2^*$. Passaseo showed in [10, 11] that this is not so: for each $1 \leq m < N-2$ he exhibited a bounded smooth domain $\mathcal{D}$ which is homotopy equivalent to the m-dimensional sphere, in which problem (1.1) has infinitely many solutions

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if $p < 2^*_{N,m} := \frac{2(N-m)}{N-m-2}$ and does not have a nontrivial solution if $p \geq 2^*_{N,m}$. Examples of domains with richer homology were recently given by Clapp, Faya and Pistoia in [3]. Wei and Yan established in [17] the existence of infinitely many positive solutions for $p = 2^*_{N,m}$ in some bounded domains. For p slightly below $2^*_{N,m}$ solutions concentrating along an m-dimensional manifold were recently obtained in [1, 4]. Note that $2^*_{N,m}$ is the critical Sobolev exponent in dimension $N-m$. It is called the $(m+1)$-st critical exponent for problem (1.1).

The purpose of this note is to exhibit unbounded domains in which this problem has the behavior described by Passaseo.

We consider the problem

$$\begin{cases} -\Delta u = |u|^{p-2}\, u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u|^2, |u|^p \in L^1(\Omega), \end{cases} \tag{1.2}$$

in a cylindrical shell

$$\Omega := \{x = (y,z) \in \mathbb{R}^{m+1} \times \mathbb{R}^{N-m-1} : a < |y| < b\}, \qquad 0 < a < b < \infty,$$

for $p > 2$.

If $m = N-1$ or $N-2$, we set $2^*_{N,m} := \infty$. First note that if $m = N-1$ then $\Omega = \{x \in \mathbb{R}^N : a < |x| < b\}$, and a well-known result by Kazdan and Warner [9] asserts that (1.2) has infinitely many radial solutions for any $p > 2$. In the other extreme case, where $m = 0$, the domain Ω is the union of two disjoint strips $(a,b)\times\mathbb{R}^{N-1}$ and $(-b,-a)\times\mathbb{R}^{N-1}$. Each of them is starshaped, so there are no solutions for $p \geq 2^*_{N,0} = 2^*$. Esteban showed in [5] that there are infinitely many solutions in $(a,b) \times \mathbb{R}^{N-1}$ if $N \geq 3$ and $p < 2^*$, and one positive solution if $N = 2$ (in fact, she considered a more general problem). These solutions are axially symmetric, i.e. $u(y,z) = u(y,|z|)$ for all $(y,z) \in \Omega$.

Here we study the remaining cases, i.e., $1 \leq m \leq N-2$. Our first result states the nonexistence of solutions other than $u = 0$, if $p \geq 2^*_{N,m}$.

Theorem 1.1. *If $1 \leq m < N-2$ and $p \geq 2^*_{N,m}$, then problem* (1.2) *does not have any nontrivial solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$.*

Our next result shows that solutions $u \neq 0$ do exist if $2 < p < 2^*_{N,m}$.

As usual, we write $O(k)$ for the group of linear isometries of $\mathbb{R}^k$ (represented by orthogonal $k \times k$-matrices). Recall that if G is a closed subgroup of $O(N)$ then a subset X of $\mathbb{R}^N$ is G-invariant if $gX = X$ for every $g \in G$, and a function $u : X \to \mathbb{R}$ is called G-invariant provided $u(gx) = u(x)$ for all $g \in G$, $x \in X$.

Note that Ω is $[O(m+1) \times O(N-m-1)]$-invariant for the obvious action given by $(g,h)(y,z) := (gy, hz)$ for all $g \in O(m+1)$, $h \in O(N-m-1)$, $y \in \mathbb{R}^{m+1}$, $z \in \mathbb{R}^{N-m-1}$.

Theorem 1.2. *(i) If $1 \leq m < N-2$ and $2 < p < 2^*_{N,m}$, then problem* (1.2) *has infinitely many $[O(m+1) \times O(N-m-1)]$-invariant solutions and one of these solutions is positive.*
(ii) If $1 \leq m = N-2$ and $2 < p < \infty$, then problem (1.2) *has a positive $[O(N-1) \times O(1)]$-invariant solution.*

In Section 2 we prove Theorem 1.1. Theorem 1.2 is proved in Section 3. We conclude the paper with a multiplicity result and an open question in Section 4.

2. A Pohožaev identity and the proof of Theorem 1.1

We prove Theorem 1.1 by adapting Passaseo's argument in [10, 11], see also [3]. The proof relies on the following special case of a Pohožaev type identity due to Pucci and Serrin [13].

For $(u,v) \in \mathbb{R} \times \mathbb{R}^N$ we set

$$\phi(u,v) := \frac{1}{2}|v|^2 - \frac{1}{p}|u|^p.$$

Lemma 2.1. *If $u \in \mathcal{C}^2(\Omega)$ satisfies $-\Delta u = |u|^{p-2}u$ in Ω then, for every $\chi \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}^N)$, the equality*

$$(\operatorname{div}\chi)\,\phi(u,\nabla u) - D\chi\,[\nabla u]\cdot\nabla u = \operatorname{div}\left[\phi(u,\nabla u)\chi - (\chi\cdot\nabla u)\nabla u\right] \tag{2.1}$$

holds true.

Proof. Put $\chi = (\chi_1, \ldots, \chi_N)$, denote the partial derivative with respect to x_k by ∂_k and let LHS and RHS denote the left- and the right-hand side of (2.1). Then

$$\text{LHS} = (\operatorname{div}\chi)\,\phi(u,\nabla u) - \sum_{j,k}\partial_k\chi_j\,\partial_j u\,\partial_k u$$

and

$$\begin{aligned}\text{RHS} = {} & (\operatorname{div}\chi)\,\phi(u,\nabla u) + \sum_{j,k}\chi_k\,\partial_j u\,\partial^2_{jk}u - |u|^{p-2}u\,\nabla u\cdot\chi \\ & - (\nabla u\cdot\chi)\Delta u - \sum_{j,k}\partial_k\chi_j\,\partial_j u\,\partial_k u - \sum_{j,k}\chi_j\,\partial_k u\,\partial^2_{jk}u \\ = {} & (\operatorname{div}\chi)\,\phi(u,\nabla u) - (\nabla u\cdot\chi)(\Delta u + |u|^{p-2}u) - \sum_{j,k}\partial_k\chi_j\,\partial_j u\,\partial_k u.\end{aligned}$$

Since $-\Delta u = |u|^{p-2}u$, the conclusion follows. ∎

Using a well-known truncation argument, we can now prove the following result.

Proposition 2.2. *Assume that* $\chi \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}^N)$ *has the following properties:*

(a) $\chi \cdot \nu$ *is bounded on* $\partial\Omega$*, where* $\nu(s)$ *is the outer unit normal at* $s \in \partial\Omega$*,*
(b) $|\chi(x)| \leq |x|$ *for every* $x \in \Omega$*,*
(c) $\operatorname{div}\chi$ *is bounded in* Ω*,*
(d) $|D\chi(x)\xi \cdot \xi| \leq |\xi|^2$ *for all* $x \in \Omega$*,* $\xi \in \mathbb{R}^N$*.*

Then every solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ *of* (1.2) *satisfies*

$$\frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 \chi \cdot \nu = -\int_{\Omega} (\operatorname{div}\chi)\, \phi(u, \nabla u) + \int_{\Omega} D\chi\, [\nabla u] \cdot \nabla u. \tag{2.2}$$

Proof. Choose $\psi \in \mathcal{C}^\infty(\mathbb{R})$ such that $0 \leq \psi(t) \leq 1$, $\psi(t) = 1$ if $|t| \leq 1$ and $\psi(t) = 0$ if $|t| \geq 2$. For each $k \in \mathbb{N}$ define

$$\psi_k(x) := \psi\left(\frac{|x|^2}{k^2}\right) \qquad \text{and} \qquad \chi^k(x) := \psi_k(x)\chi(x).$$

Note that there is a constant $c_0 > 0$ such that

$$|x|\,|\nabla\psi_k(x)| \leq c_0 \qquad \text{for all } x \in \mathbb{R}^N,\ k \in \mathbb{N}. \tag{2.3}$$

Next, choose a sequence of bounded smooth domains $\Omega_k \subset \Omega$ such that

$$\Omega_k \supset \Omega \cap \overline{B_{2k}(0)}. \tag{2.4}$$

Integrating (2.1) with $\chi := \chi^k$ in Ω_k and using the divergence theorem and Lemma 2.1 we obtain

$$\begin{gathered}\int_{\Omega_k} \left(\operatorname{div}\chi^k\right) \phi(u, \nabla u) - \int_{\Omega_k} D\chi^k\, [\nabla u] \cdot \nabla u = \\ \int_{\partial\Omega_k} \left[\phi(u, \nabla u)\left(\chi^k \cdot \nu^k\right) - (\chi^k \cdot \nabla u)\left(\nabla u \cdot \nu^k\right)\right],\end{gathered}$$

where ν^k is the outer unit normal to Ω_k. Property (2.4) implies that $\chi^k = 0$ in $\overline{\Omega \smallsetminus \Omega_k}$, so we may replace Ω_k by Ω, $\partial\Omega_k$ by $\partial\Omega$ and ν^k by ν in the previous identity. Moreover, since $u = 0$ on $\partial\Omega$, we have that

$$\nabla u = (\nabla u \cdot \nu)\,\nu \ \text{ on } \partial\Omega.$$

Therefore,

$$\begin{aligned}&\int_{\Omega} \left(\operatorname{div}\chi^k\right) \phi(u, \nabla u) - \int_{\Omega} D\chi^k\, [\nabla u] \cdot \nabla u = \\ &\int_{\partial\Omega} \left[\phi(u, \nabla u)\left(\chi^k \cdot \nu\right) - (\chi^k \cdot \nabla u)\,(\nabla u \cdot \nu)\right] = \\ &\int_{\partial\Omega} \left[\phi(u, \nabla u) - |\nabla u|^2\right]\left(\chi^k \cdot \nu\right) = -\frac{1}{2}\int_{\partial\Omega} |\nabla u|^2\, \psi_k(x)\, (\chi \cdot \nu)\,.\end{aligned} \tag{2.5}$$

Since $\operatorname{div}\chi^k = \psi_k \operatorname{div}\chi + \nabla\psi_k \cdot \chi$, using (2.3) and properties (b) and (c) we obtain

$$\left|\operatorname{div}\chi^k\right| \leq |\operatorname{div}\chi| + |\nabla\psi_k|\,|\chi| \leq |\operatorname{div}\chi| + c_0 \leq c_1 \quad \text{in } \Omega. \tag{2.6}$$

Similarly, since

$$D\chi^k(x)\xi\cdot\xi = \psi_k(x)D\chi(x)\xi\cdot\xi + (\nabla\psi_k\cdot\xi)\,(\chi\cdot\xi)\,,$$

property (d) yields

$$\left|D\chi^k(x)\xi\cdot\xi\right| \leq (1+c_0)\,|\xi|^2 \qquad \text{for all } x\in\Omega,\ \xi\in\mathbb{R}^N. \tag{2.7}$$

Inequalities (2.6), (2.7) and property (a) allow us to apply Lebesgue's dominated convergence theorem to the left- and the right-hand side of (2.5) to obtain

$$\int_\Omega (\operatorname{div}\chi)\,\phi(u,\nabla u) - \int_\Omega D\chi\,[\nabla u]\cdot\nabla u = -\frac{1}{2}\int_{\partial\Omega}|\nabla u|^2\,(\chi\cdot\nu)\,,$$

as claimed. □

Proof of Theorem 1.1. Let $\varphi(t) = \frac{1}{m+1}\left[1-(\frac{a}{t})^{m+1}\right]$ be the solution to the boundary value problem

$$\begin{cases} \varphi'(t)t + (m+1)\varphi(t) = 1, & t\in(0,\infty),\\ \varphi(a) = 0. \end{cases}$$

Define

$$\chi(y,z) := (\varphi(|y|)y, z). \tag{2.8}$$

Then, if ν denotes the outer unit normal on $\partial\Omega$,

$$(\chi\cdot\nu)\,(y,z) = \begin{cases} 0 & \text{if } |y| = a,\\ \frac{1}{m+1}\left[1-(\frac{a}{b})^{m+1}\right]b & \text{if } |y| = b. \end{cases} \tag{2.9}$$

So property (a) of Proposition 2.2 holds. Clearly, (b) holds. Now,

$$\operatorname{div}\chi(y,z) = \left[\varphi'(|y|)\,|y| + (m+1)\varphi(|y|)\right] + N - m - 1 = N - m. \tag{2.10}$$

In particular, (c) holds. To prove (d) notice that χ is $O(m+1)$-equivariant, i.e.

$$\chi(gy,z) = g\chi(y,z) \qquad \text{for every } g\in O(m+1).$$

Therefore, $g\circ D\chi(y,z) = D\chi(gy,z)\circ g$ and, hence,

$$\langle D\chi\,(y,z)\,[\xi]\,,\xi\rangle = \langle g\,(D\chi\,(y,z)\,[\xi])\,,g\xi\rangle = \langle D\chi\,(gy,z)\,[g\xi], g\xi\rangle$$

for all $\xi\in\mathbb{R}^N$. Thus, it suffices to show that the inequality (d) holds for $y = (t,0,\ldots,0)$ with $t\in(a,b)$. A straightforward computation shows that, for such y, $D\chi(y)$ is a diagonal matrix whose diagonal entries are $a_{11} =$

$1-m\varphi(t)$, $a_{jj}=\varphi(t)$ for $j=2,\ldots,m+1$, and $a_{jj}=1$ for $j=m+2,\ldots,N$. Since $a_{jj}\in(0,1]$,

$$0<\langle D\chi(y,z)[\xi],\xi\rangle\leq|\xi|^2\quad\text{for all }\xi\in\mathbb{R}^N\smallsetminus\{0\}\tag{2.11}$$

and (d) follows. From (2.9), (2.2), (2.11) and (2.10) we obtain

$$\begin{aligned}0&<\frac{1}{2}\int_{\partial\Omega}|\nabla u|^2\,\chi\cdot\nu=-\int_\Omega(\operatorname{div}\chi)\,\phi(u,\nabla u)+\int_\Omega D\chi\,[\nabla u]\cdot\nabla u\\&\leq(N-m)\int_\Omega\left[\frac{1}{p}|u|^p-\frac{1}{2}|\nabla u|^2\right]+\int_\Omega|\nabla u|^2\\&=(N-m)\left(\frac{1}{p}-\frac{1}{2}+\frac{1}{N-m}\right)\int_\Omega|\nabla u|^2.\end{aligned}$$

The first (strict) inequality follows from the unique continuation property [8, 7]. This immediately implies that $p<2^*_{N,m}$. □

3. The proof of Theorem 1.2

An $O(m+1)$-invariant function $u(y,z)=v(|y|,z)$ solves problem (1.2) if and only if $v=v(r,z)$ solves

$$\begin{cases}-\Delta v-\frac{m}{r}\frac{\partial v}{\partial r}=|v|^{p-2}v & \text{in }(a,b)\times\mathbb{R}^{N-m-1}=:\mathcal{S},\\ v=0 & \text{on }\{a,b\}\times\mathbb{R}^{N-m-1}=\partial\mathcal{S},\end{cases}\tag{3.1}$$

and $|\nabla v|^2,|v|^p\in L^1(\mathcal{S})$. Problem (3.1) can be rewritten as

$$-\operatorname{div}(r^m\nabla v)=r^m|v|^{p-2}v\quad\text{in }\mathcal{S},\qquad v=0\quad\text{on }\partial\mathcal{S}.\tag{3.2}$$

By Poincaré's inequality (see Lemma 3 in [5]) and since $a<r<b$, the norms

$$\|v\|_m:=\left(\int_{\mathcal{S}}r^m|\nabla v|^2\right)^{1/2}\qquad\text{and}\qquad|v|_{m,p}:=\left(\int_{\mathcal{S}}r^m|v|^p\right)^{1/p}\tag{3.3}$$

are equivalent to those of $H^1_0(\mathcal{S})$ and $L^p(\mathcal{S})$ respectively.

Consider the functional $I(v):=\|v\|_m^2$ restricted to

$$M:=\{v\in H^1_0(\mathcal{S}):|v|_{m,p}=1\}.$$

Then M is a C^2-manifold, and v is a critical point of $I|_M$ if and only if $v\in H^1_0(\mathcal{S})$ and $\|v\|_m^{2/(p-2)}v$ is a nontrivial solution to (3.2). Note that $I|_M$ is bounded below by a positive constant.

Proof of Theorem 1.2 (i). Assume that $1\leq m<N-2$ and $2<p<2^*_{N,m}$. Set $G:=O(N-m-1)$ and denote by $H^1_0(\mathcal{S})^G$ and $L^p(\mathcal{S})^G$ the subspaces of $H^1_0(\mathcal{S})$ and $L^p(\mathcal{S})$ respectively, consisting of functions v such that $v(r,gz)=v(r,z)$ for all $g\in G$. Esteban and Lions showed in [6] that, for these values of m and p, $H^1_0(\mathcal{S})^G$ is compactly embedded in $L^p(\mathcal{S})^G$ (see also Theorem

1.24 in [18]). So $H_0^1(\mathcal{S})^G$ is compactly embedded in $L^p(\mathcal{S})^G$ for the norms (3.3) as well.

Let

$$M^G := \{v \in H_0^1(\mathcal{S})^G : |v|_{m,p} = 1\}.$$

It follows from the principle of symmetric criticality [18, Theorem 1.28] that the critical points of $I|_{M^G}$ are also critical points of $I|_M$. The manifold M^G is radially diffeomorphic to the unit sphere in $H_0^1(\mathcal{S})^G$, so its Krasnoselskii genus is infinite. A standard argument, using the compactness of the embedding $H_0^1(\mathcal{S})^G \hookrightarrow L^p(\mathcal{S})^G$ for the norms (3.3), shows that $I|_{M^G}$ satisfies the Palais-Smale condition. Hence $I|_{M^G}$ has infinitely many critical points (see e.g. Theorem II.5.7 in [15]). It can also be shown by a well-known argument that the critical values of $I|_{M^G}$ tend to infinity (see e.g. Proposition 9.33 in [14]).

It remains to show that (3.2) has a positive solution. The argument is again standard: since $I|_{M^G}$ satisfies the Palais-Smale condition,

$$c_0^G := \inf\{I(v) : v \in M^G\}$$

is attained at some v_0. Since $I(v) = I(|v|)$ and $|v| \in M^G$ if $v \in M^G$, we have that $I(|v_0|) = c_0^G$ and we may assume $v_0 \geq 0$. The maximum principle applied to the corresponding solution u_0 of (1.2) implies $u_0 > 0$. $\square$

If $m = N - 2$, then $G = O(1)$ and it is easy to see that the space $H_0^1(\mathcal{S})^G$ is not compactly embedded in $L^p(\mathcal{S})^G$. So part (ii) of Theorem 1.2 requires a different argument.

Proof of Theorem 1.2 (ii). Assume that $1 \leq m = N - 2$ and $2 < p < \infty$. We shall show that

$$c_0 := \inf\{I(v) : v \in M\}$$

is attained. Clearly, a minimizing sequence (v_n) is bounded, so we may assume that $v_n \rightharpoonup v$ weakly in $H_0^1(\mathcal{S})$. According to P.-L. Lions' lemma [18, Lemma 1.21] either $v_n \to 0$ strongly in $L^p(\mathcal{S})$, which is impossible because $v_n \in M$, or there exist $\delta > 0$ and $(r_n, z_n) \in [a, b] \times \mathbb{R}$ such that, after passing to a subsequence if necessary,

$$\int_{B_1(r_n,z_n)} v_n^2 \geq \delta. \tag{3.4}$$

Here $B_1(r_n, z_n)$ denotes the ball of radius 1 and center at (r_n, z_n). Since the problem is invariant with respect to translations along the z-axis, replacing $v_n(r, z)$ by $v_n(r, z + z_n)$, we may assume the center of the ball above is $(r_n, 0)$. It follows that for this - translated - sequence the weak limit v cannot be zero due to (3.4) and the compactness of the embedding of $H_0^1(\mathcal{S})$

in $L^2_{loc}(\mathcal{S})$. Passing to a subsequence once more, we have that $v_n(x) \to v(x)$ a.e. It follows from the Brezis-Lieb lemma [18, Lemma 1.32] that

$$1 = |v_n|^p_{m,p} = \lim_{n\to\infty} |v_n - v|^p_{m,p} + |v|^p_{m,p}.$$

Using this identity and the definition of c_0 we obtain

$$\begin{aligned} c_0 &= \lim_{n\to\infty} \|v_n\|^2_m = \lim_{n\to\infty} \|v_n - v\|^2_m + \|v\|^2_m \geq c_0 \left(\lim_{n\to\infty} |v_n - v|^2_{m,p} + |v|^2_{m,p} \right) \\ &= c_0 \left((1 - |v|^p_{m,p})^{2/p} + (|v|^p_{m,p})^{2/p} \right) \geq c_0 (1 - |v|^p_{m,p} + |v|^p_{m,p})^{2/p} = c_0. \end{aligned}$$

Since $v \neq 0$, it follows that $|v_n - v|_{m,p} \to 0$ and $|v|_{m,p} = 1$. So $v \in M$ and, as $c_0 = \lim_{n\to\infty} I(v_n) \geq I(v)$, we must have $I(v) = c_0$.

So the infimum is attained at v and using the moving plane method [18, Appendix C], we may assume, after translation, that $v(r, -z) = v(r, z)$, i.e. $v \in H^1_0(\mathcal{S})^{O(1)}$. As in the preceding proof, replacing v by $|v|$, we obtain a positive solution. □

4. Further solutions and an open question

If $1 \leq m = N - 2$ and $p \in (2, 2^*_{N,m})$, the method we have used to prove Theorem 1.2 only guarantees the existence of two solutions to problem (1.2), one positive and one negative, up to translations along the z-axis. However, if $p \in (2, 2^*)$, then it is possible to show that there are infinitely many solutions, which are not radial in y, but have other prescribed symmetry properties.

Write $y = (y^1, y^2) \in \mathbb{R}^2 \times \mathbb{R}^{m-1} \equiv \mathbb{R}^{m+1}$ and identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Following [16], we denote by G_k, $k \geq 3$, the subgroup of $O(2)$ generated by two elements α, β which act on $\mathbb{C}$ by

$$\alpha y^1 := e^{2\pi i/k} y^1, \qquad \beta y^1 := e^{2\pi i/k} \overline{y^1},$$

i.e. α is the rotation in $\mathbb{C}$ by the angle $2\pi/k$ and β is the reflection in the line $y^1_2 = \tan(\pi/k) y^1_1$, where $y^1 = y^1_1 + \mathrm{i} y^1_2 \in \mathbb{C}$. Observe that α, β satisfy the relations $\alpha^k = \beta^2 = e$, $\alpha\beta\alpha = \alpha$. Let G_k act on $\mathbb{R}^N$ by $gx = (gy^1, y^2, z)$.

Theorem 4.1. *If $1 \leq m \leq N - 2$ and $2 < p < 2^*$ then, for each $k \geq 3$, problem (1.2) has a solution u_k which satisfies*

$$u_k(x) = \det(g) u_k(g^{-1}x) \qquad \textit{for all } g \in G_k, \tag{4.1}$$

and $u_k \neq u_j$ if $k \neq j$.

Proof. Since the approach is taken from [16], we give only a brief sketch of the proof here and refer to Section 2 of [16] for more details.

The group G_k acts on $H_0^1(\Omega)$ by

$$(gu)(x) := \det(g)u(g^{-1}x),$$

where $\det(g)$ is the determinant of g. Let

$$H_0^1(\Omega)^{G_k} := \{u \in H_0^1(\Omega) : u(gx) = \det(g)u(g^{-1}x) \text{ for all } g \in G_k\}$$

be the fixed point space of this action, and define $I(u) := \int_\Omega |\nabla u|^2$ and

$$M^{G_k} := \{u \in H_0^1(\Omega)^{G_k} : |u|_p = 1\}.$$

By the principle of symmetric criticality the critical points of $I|_{M^{G_k}}$ are nontrivial solutions to problem (1.2) which satisfy (4.1). Now we can see as in the proof of part (ii) of Theorem 1.2 that there exists a minimizer u_k for I on the manifold M^{G_k}. Moreover, we may assume that u_k has exactly $2k$ nodal domains, see Corollary 2.7 in [16]. So in particular, $u_k \neq u_j$ if $k \neq j$. □

The question whether problem (1.2) has infinitely many solutions when $1 \leq m = N-2$ and $p \in [2^*, 2^*_{N,m})$ remains open. We believe that the answer is yes, but the proof would require different methods.

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